Tutorial Note X

Exercise 0.1

1. Calculate the Fourier coefficients of $1_{[0,1]}$ on $[-1,1]$.
2. Calculate the Fourier coefficients of $|x|$ on $[-1,1]$.
3. Calculate $\sum_{n=1}^{\infty} 1/n^2$.

Proof.

1. If $n \neq 0$,

$$\begin{aligned}\hat{f}(n) &= \frac{1}{2}\int_{-1}^{1} 1_{[0,1]}(x)\mathrm{e}^{-\mathrm{i}n\pi x}\,\mathrm{d}x \\ &= \frac{1}{2}\int_{0}^{1} \mathrm{e}^{-\mathrm{i}n\pi x}\,\mathrm{d}x \\ &= \frac{1-(-1)^n}{2\mathrm{i}n\pi}.\end{aligned}$$

   If $n = 0$,

$$\begin{aligned}\hat{f}(0) &= \frac{1}{2}\int_{-1}^{1} 1_{[0,1]}(x)\,\mathrm{d}x \\ &= \frac{1}{2}.\end{aligned}$$

2. If $n \neq 0$,

$$\begin{aligned}\hat{f}(n) &= \frac{1}{2}\int_{-1}^{1} |x|\mathrm{e}^{-\mathrm{i}n\pi x}\,\mathrm{d}x \\ &= \frac{1}{2}\left(\int_{0}^{1} x\mathrm{e}^{-\mathrm{i}n\pi x}\,\mathrm{d}x + \int_{-1}^{0} (-x)\mathrm{e}^{-\mathrm{i}n\pi x}\,\mathrm{d}x\right) \\ &= \frac{1}{2}\left(\int_{0}^{1} x\mathrm{e}^{-\mathrm{i}n\pi x}\,\mathrm{d}x + \int_{0}^{1} x\mathrm{e}^{\mathrm{i}n\pi x}\,\mathrm{d}x\right) \\ &= \int_{0}^{1} x\cos n\pi x\,\mathrm{d}x \\ &= -\int_{0}^{1} \frac{\sin n\pi x}{n\pi}\,\mathrm{d}x \\ &= \frac{(-1)^n - 1}{n^2\pi^2}.\end{aligned}$$

If $n = 0$,

$$\hat{f}(0) = \int_0^1 x \cos 0\pi x \, \mathrm{d}x$$
$$= \frac{1}{2}.$$

3. By 2,

$$|x| = \frac{1}{2} + \sum_{n \neq 0} \frac{(-1)^n - 1}{n^2 \pi^2} \mathrm{e}^{\mathrm{i}n\pi x}.$$

Since $|x|$ is Lipschitz continuous, this equality holds. Letting $x = 0$, we have

$$0 = \frac{1}{2} - 2 \sum_{n \in 2\mathbb{Z}+1} \frac{1}{n^2 \pi^2}.$$

So

$$\sum_{n \in 2\mathbb{Z}+1} \frac{1}{n^2} = \frac{\pi^2}{4}.$$

Since

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \sum_{n \in 2\mathbb{Z} \cap \mathbb{N}^+} \frac{1}{n^2} + \sum_{n \in 2\mathbb{Z}+1 \cap \mathbb{N}^+} \frac{1}{n^2}$$
$$= \frac{1}{4} \sum_{n=1}^{\infty} \frac{1}{n^2} + \frac{\pi^2}{8},$$

we have

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

Alternatively, we could use 1 and the Parseval identity to calculate $\sum_{n=1}^{\infty} 1/n^2$. In fact,

$$\int_{-1}^{1} 1_{[0,1]}^2 = \left(\frac{1}{\sqrt{2}}\right)^2 + \sum_{n \in 2\mathbb{Z}+1} \left(\frac{\sqrt{2}}{n\pi}\right)^2,$$

yielding that

$$\sum_{n \in 2\mathbb{Z}+1} \frac{1}{n^2} = \frac{\pi^2}{4}.$$

□

Remark 0.1

There are a lot of methods to solve the Basel problem $\sum_{n=1}^{\infty} 1/n^2$, except Fourier series, such as generating functions, residues, and the Poisson summation formula. Here if we start with the Basel problem, how to find the function to calculate Fourier coefficients? In fact, there are not only one function available. For example, if we want to use the Parseval identity, we need to calculate $(1/n)^{\vee}$. By the formula

$$\hat{f'}(n) = \mathrm{i}n\pi \hat{f}(n),$$

and

$$\delta(0)\widehat{\ }=\frac{1}{2},$$

a candidate is the above $1_{[0,1]}$ that is an integral of $\delta(0)$. In fact, $1_{[0,1]}$ is not a genuine antiderivative of $\delta(0)$, and $1'_{[0,1]}=\delta(0)-\delta(1)$.

Exercise 0.2

Derive Poisson's formula for the Dirichlet problem of the Laplace equation on the disk of $\mathbb{R}^2$ by Fourier series.

Proof. For the Dirichlet problem:

$$\begin{cases}\Delta u=0 & \text{in } \mathbb{D};\\ u=f & \text{on } \partial\mathbb{D},\end{cases}$$

we use the method of separation of variables to solve it. First we find special solutions with the form $R(r)\Theta(\theta)$. By the polar coordinate expression of the Laplacian, we have

$$R''\Theta+\frac{1}{r}R'\Theta+\frac{1}{r^2}R\Theta''=0.$$

So

$$\frac{r^2R''+rR'}{R}=-\frac{\Theta''}{\Theta}=\lambda.$$

Θ satisfies the periodic condition $\Theta(0)=\Theta(2\pi)$. Thus,

$$\Theta_n(\theta)=\mathrm{e}^{\mathrm{i}n\theta},$$

where $n\in\mathbb{Z}$, and

$$R_n(r)=r^{\pm|n|}.$$

We choose $R_n(r)=r^{|n|}$ to avoid the singularity at the origin. Now if we use Fourier series to write $f(\theta)$ as

$$\sum_n \hat{f}(n)\mathrm{e}^{\mathrm{i}n\theta},$$

then

$$u(r,\theta)=\sum_n r^{|n|}\hat{f}(n)\mathrm{e}^{\mathrm{i}n\theta}$$

is a candidate of the solution to the Dirichlet problem. Next we simplify the u.

$$\begin{aligned}u(r,\theta)&=\frac{1}{2\pi}\sum_n r^{|n|}\int_0^{2\pi}f(\varphi)\mathrm{e}^{-\mathrm{i}n\varphi}\,\mathrm{d}\varphi\mathrm{e}^{\mathrm{i}n\theta}\\&=\frac{1}{2\pi}\int_0^{2\pi}\sum_n r^{|n|}f(\varphi)\mathrm{e}^{\mathrm{i}n(\theta-\varphi)}\,\mathrm{d}\varphi\\&=\frac{1}{2\pi}\int_0^{2\pi}f(\varphi)\left(1+\frac{r\mathrm{e}^{\mathrm{i}(\theta-\varphi)}}{1-r\mathrm{e}^{\mathrm{i}(\theta-\varphi)}}++\frac{r\mathrm{e}^{-\mathrm{i}(\theta-\varphi)}}{1-r\mathrm{e}^{-\mathrm{i}(\theta-\varphi)}}\right)\mathrm{d}\varphi\end{aligned}$$

$$= \frac{1}{2\pi} \int_0^{2\pi} f(\varphi) \frac{1 - r^2}{1 - 2r\cos(\theta - \varphi) + r^2} \, \mathrm{d}\varphi.$$

Now we obtain Poisson's formula. □